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1984 J. Phys. A: Math. Gen. 17 3343

(http://iopscience.iop.org/0305-4470/17/17/012)

J. Phys. A: Math. Gen. 17 (1984) 3343-3351. Printed in Great Britain

Directed percolation: A Monte Carlo approach

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Received 13 April 1984, in final form 9 July 1984

Abstract. This paper describes a Monte Carlo study of two-dimensional directed site percolation. We show that the anisotropic behaviour resulting from the imposition of a directional bias must be accounted for in the finite-size scaling analysis of the Monte Carlo data if reliable results are to be obtained. Using suitably modified finite-size scaling we obtain estimates for the percolation threshold P_C and the exponents ν_P, ν_T and β which are in good agreement with results obtained by other methods.

1. Introduction

Percolation models have become increasingly popular in recent years due to the wide variety of possible applications and the theoretical interest in the percolation threshold as a geometrical phase transition (for recent reviews and extensive references see Stauffer 1979, Essam 1980, Deutscher *et al* 1983). Among the many recent developments there has been a great deal of interest in studying the effects of imposing a global directional bias in these models (for an excellent introduction see Kinzel 1983). The introduction of a preferred direction strongly affects the large-scale behaviour and leads to anisotropic scaling and direction-dependent critical properties.

An example of directed site percolation is shown in figure 1. Here bonds only occur between nearest-neighbour occupied sites, which are present with probability P, and percolation is only permitted in the direction of the arrows. In order to discuss percolation across a lattice it is necessary to specify a set of sources (typically the initial boundary row as in figure 1). We then imagine that a fluid is introduced at the source sites and flows along the allowed bonds in the permitted directions; those sites 'wetted' by the fluid define the cluster of sites connected to the source (in figure 1 fragments (A), (B) and (C) are not connected to the source sites). Perhaps the most important new feature is the appearance of two different correlation length exponents ν_P and ν_T, which characterise the divergence of the correlation lengths ξ_P and ξ_T in the preferred and transverse directions respectively. Below the percolation threshold only finite-sized clusters occur and these become increasingly anisotropic in shape as the threshold is approached, growing more rapidly in the preferred direction than in the transverse directions. The behaviour is described in the following way:

$$\xi_P \sim \xi_T^{\theta} \sim (P_C - P)^{-\nu_P} \sim (P_C - P)^{-\nu_T \theta} \tag{1.1}$$

where P_C is the percolation threshold and $\theta = \nu_P/\nu_T$ is the anisotropy exponent. The existence of two characteristic length scales is reminiscent of the situation in critical dynamics, where there is both a characteristic length and a time scale; indeed it is

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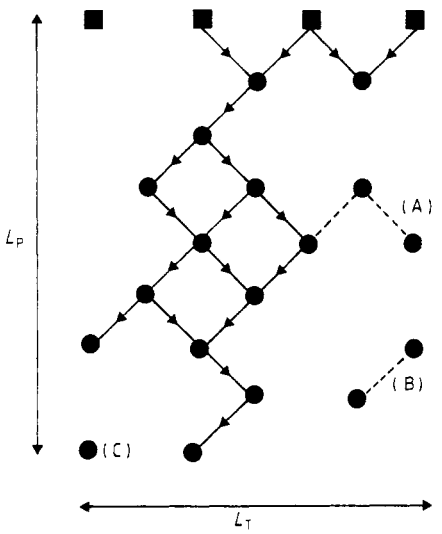

Figure 1. An example of directed site percolation on a square lattice. The arrows indicate the allowed directions of flow along the bonds.

often useful to think of the preferred direction as being 'time-like' and to regard θ as being similar to the dynamic critical exponent z.

Cardy and Sugar (1980) have shown that directed percolation can be mapped onto a Reggeon field theory which models the creation, propagation and destruction of a cascade of elementary particles. This is a particular example of the deeper connection between directed percolation and branching Markov processes (see for example Grassberger and de la Torre 1979), which is useful in describing diffusion–reaction types of processes. As Kinzel (1983) has pointed out, 2D-directed bond percolation can be thought of as an autocatalytic contact reaction in one dimension. The directed percolation cluster configurations are equivalent to the trajectories of a diffusing–reacting particle system in one space and one time dimension, where particles may diffuse, split, recombine or disappear. The lifetime of the population of particles is equivalent to ξ_P in directed percolation, while the spatial extent of the population is equivalent to ξ_T.

Critical thresholds and exponents for directed percolation have been calculated by means of series expansions (Blease 1977, Essam and de'Bell 1981, de'Bell and Essam 1983a, b), transfer matrix calculations (Kinzel and Yeomans 1981), and Monte Carlo simulations (Kertesz and Vicsek 1980, Dhar and Barma 1981). Universal critical properties have also been calculated by exploiting the connection with reaction models which belong to the same universality class, again techniques such as ε or series expansions (see Cardy and Sugar (1980) for references) or Monte Carlo simulations (Grassberger and de la Torre 1979) have been used. The Monte Carlo simulations of the directed percolation models gave disappointing results in comparison with the other techniques and with the Monte Carlo simulations of reaction models. The aim of this present paper is to attempt to redress this situation by improving on the previous Monte Carlo work.

More specifically, we will focus our attention on directed site percolation on the 2D square lattice, as illustrated in figure 1. We estimate the percolation threshold, P_C,

and the correlation length exponents, ν_P and ν_T, by extending the usual Monte Carlo finite-size scaling techniques (see for instance Reynolds *et al* 1978, 1980) to cope with the anisotropy in the problem. We also show that the earlier simulations of Kertesz and Vicsek (1980) failed to handle this anisotropy correctly, although this is not immediately apparent in their original data. Finally, following the method used by Dhar and Barma (1981), we estimate the value of the exponent β, which characterises the non-analytic behaviour of the fraction of sites belonging to the infinite cluster as the percolation threshold is approached from above.

2. Method and Results

We begin by reviewing the standard Monte Carlo finite-size scaling method, which is used to estimate the percolation threshold and correlation-length exponent for ordinary percolation (Reynolds *et al* 1978, 1980, see also Stauffer 1981). The method utilises an iterative Monte Carlo scheme to determine threshold probabilities for a series of realisations of finite lattices of various sizes and then uses finite-size scaling to extrapolate the results to an infinite lattice. The difficulties introduced by the presence of two length scales only affect the finite-size scaling theory aspects of the method. Firstly we take a finite lattice of size L^d and assign a random number r_i from the interval $(0, 1)$ to each site i; we call this a lattice realisation. We find the percolation threshold probability for this particular realisation by a binary search method. We let $P_C(L, 0)$ be an initial estimate for this threshold probability and we designate those sites where $r_i < P_C(L, 0)$ as being occupied, we then check to see whether a percolating cluster is present or not at this particular estimate. If a percolating cluster is present we try a new estimate $P_C(L, 1) < P_C(L, 0)$, if not we try $P_C(L, 1) > P_C(L, 0)$ and again we check for percolation at the new estimate. This procedure is repeated until we have two successive estimates $P_C(L, N)$ and $P_C(L, N+1)$ which bound an interval containing the true threshold value. Having found this interval we can then locate the threshold to any required accuracy by successive binary chopping of the interval containing the threshold. Finally by repeating the whole process many times, for different random number sequences, we obtain a series of estimates for $P_C(L)$. The average value of these estimates, $\langle P_C(L)\rangle$, is taken as an estimate for the threshold, and the spread in the estimates determines the correlation-length exponent, ν, through finite-size scaling. For ordinary percolation we have

$$\sigma(L) = (\langle P_C(L)^2\rangle - \langle P_C(L)\rangle^2)^{1/2} \sim L^{-1/\nu} \tag{2.1}$$

(see Reynolds *et al* 1978, 1980). This result follows from the finite-size scaling hypothesis for the correlation length, namely

$$\xi(L; P) \approx \xi(\infty; P)\, Y(L/\xi(\infty; P)) \tag{2.2}$$

where the scaling function $Y(x)$ approaches the value 1 for $x \gg 1$, so that (2.2) reduces to an identity in the limit $L \to \infty$, and behaves like Ax for $0 < x \ll 1$, so that ξ_L remains finite as ξ_∞ diverges in the limit $P \to P_C$.

A natural generalisation for the directed percolation counterparts is

$$\begin{aligned}\xi_P(L_P, L_T; P) &\approx \xi_P(\infty, \infty; P) F(L_P/\xi_P(\infty, \infty; P),\ L_T/\xi_T(\infty, \infty; P)) \\ &\approx \xi_P(\infty, \infty; P) G(L_P/\xi_P(\infty, \infty; P),\ L_T^\theta/\xi_P(\infty, \infty; P))\end{aligned} \tag{2.3}$$

and similarly for $\xi_T(L_P, L_T; P)$, where L_P and L_T denote the extent of the lattice in the preferred and transverse directions respectively. Although the above generalisation appears reasonable one should note the cautionary example of the directed self-avoiding walk where a third length enters into the finite-size scaling theory (Szpilka and Privman 1983). Equation (2.3) is termed a strong scaling hypothesis and has been successfully employed in transfer matrix calculations by Kinzel and Yeomans (1981). In these calculations $\xi_P(\infty, L_T; P)$ is calculated on strips of width L_T (i.e. $L_P = \infty$, L_T finite) and (2.3) reduces to a single variable function which can be exploited by the usual methods of phenomenological scaling. However, in the Monte Carlo approach the situation is different as L_P and L_T are both finite.

In the Monte Carlo approach we investigate finite lattices of size L_P by L_T. We estimate the percolation threshold by finding the smallest value of P for which there is a cluster that spans the lattice in the preferred direction; this is achieved by using the binary search procedure outlined above. Since the finite-size clusters become increasingly anisotropic in shape, as the threshold is approached from below, we might expect to find a regime where $G(x, y)$ is controlled by its first argument. We expect to see spanning clusters when ξ_P becomes comparable with L_P, so by choosing $L_P = L_T = L$ sufficiently large we should find that the first argument of G is of order unity while the second argument is very large when P is near $P_C(L)$, then by analogy with the behaviour of $Y(x)$ in (2.2) we expect the behaviour of $G(x, y)$ to be controlled by x. If such a regime exists then (2.1) should apply with ν replaced by ν_P. This is essentially the approach taken by Kertesz and Vicsek (1980) for the directed bond percolation problem on the square lattice, although the scaling assumptions were not explicitly mentioned.

We have followed the same approach for the site problem and our results for a series of $L \times L$ lattices are given in table 1. The numbers in parentheses give the approximate statistical uncertainty in the last digit(s) quoted (as taken from the sample-to-sample variation in the estimates). In figure 2 we have plotted $\log[\sigma(L)^{-1}]$ against $\log(L)$, which according to (2.1) should give a straight line with slope ν_P^{-1}. Our results appear to support this scaling hypothesis but from the slope we obtain

$$\nu_P = 1.53(2) \tag{2.4}$$

which compares very unfavourably with other estimates of ν_P by series methods etc. If this scaling hypothesis is valid then we expect

$$|P_C(\infty) - \langle P_C(L)\rangle| \sim L^{-1/\nu_P}. \tag{2.5}$$

Table 1. Results for isotropic finite lattices.

L	Number of realisations	$\langle P_C(L)\rangle$	$\sigma(L)$
32	204 800	0.638 09(7)	0.031 32(5)
64	204 800	0.655 66(5)	0.019 67(3)
128	51 200	0.669 40(11)	0.012 43(5)
256	12 800	0.679 51(7)	0.007 85(7)
512	3 200	0.686 98(9)	0.005 02(9)
1024	800	0.692 32(11)	0.003 21(11)
2048	200	0.696 25(15)	0.002 07(15)
4096	50	0.699 22(16)	0.001 12(16)

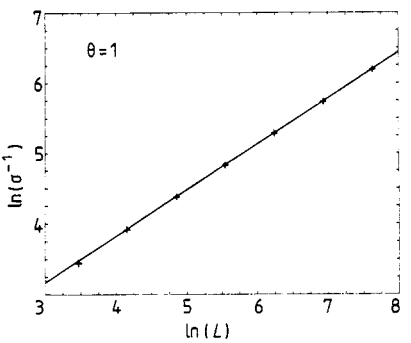

Figure 2. A plot of $\log(\sigma^{-1})$ against $\log(L)$ for a series of $L \times L$ lattices.

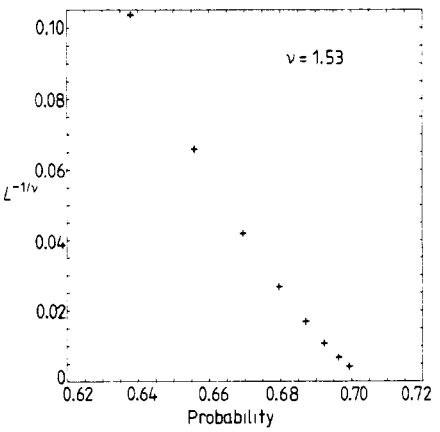

Figure 3. A plot of L^{-1/ν_P} against $\langle P_C(L)\rangle$ for a series of $L \times L$ lattices.

Substituting from (2.4) into (2.5) and plotting $\langle P_C(L)\rangle$ against L^{-1/ν_P} we do not obtain the expected straight line, see figure 3. This suggests that either L is too small to see scaling behaviour or that the naive scaling hypothesis is not valid. The latter conclusion seems more likely, particularly as the results are further from the accepted values than those of Kertesz and Vicsek (1980) despite the larger lattices used.

In order to obtain the correct scaling behaviour we must handle the anisotropy more carefully. We use (2.3) to write

$$\frac{\xi_P(b_P L_P, b_T L_T; P')}{\xi_P(L_P, L_T; P)} = \frac{\xi_P(\infty,\infty; P')G[b_P L_P/\xi_P(\infty,\infty; P'), b_T^{\theta} L_T^{\theta}/\xi_P(\infty,\infty; P')]}{\xi_P(\infty,\infty; P)G[L_P/\xi_P(\infty,\infty; P), L_T^{\theta}/\xi_P(\infty,\infty; P)]}. \tag{2.6}$$

If for a given L_P, L_T, P and rescaling factor b_P we choose P' and b_T so that

$$\xi_P(\infty,\infty; P') = b_P \xi_P(\infty,\infty; P) \tag{2.7}$$

$$b_T = b_P^{1/\theta} \tag{2.8}$$

then the unknown function G drops out of (2.6) and we have

$$\xi_P(b_P L_P, b_P^{1/\theta} L_T; P')/\xi_P(L_P, L_T; P) = b_P. \tag{2.9}$$

If P is taken to be $\langle P_C(L_P, L_T)\rangle$ then (2.9) will be satisfied when $P' = \langle P_C(b_P L_P, b_P^{1/\theta} L_T)\rangle$, since the finite lattice threshold is attained when the correlation length is comparable to the system size. When P and P' take on these values then (2.7) will also be satisfied and using (1.1) we have

$$(P_C(\infty,\infty) - \langle P_C(L_P, L_T)\rangle)^{\nu_P}/(P_C(\infty,\infty) - \langle P_C(b_P L_P, b_P^{1/\theta} L_T)\rangle)^{\nu_P} = b_P. \tag{2.10}$$

Rescaling by a factor b_P in the preferred direction and by $b_T = b_P^{1/\theta}$ in the transverse direction we use (2.10) to obtain results similar to (2.1) and (2.5)

$$\sigma(b_P L_P, b_P^{1/\theta} L_T) = (\langle P_C(b_P L_P, b_P^{1/\theta} L_T)^2\rangle - \langle P_C(b_P L_P, b_P^{1/\theta} L_T)\rangle)^{1/2} \approx (b_P L_P)^{1/\nu_P} \tag{2.11}$$

$$|P_C(\infty,\infty) - \langle P_C(b_P L_P, b_P^{1/\theta} L_P)\rangle| \approx (b_P L_P)^{-1/\nu_P}. \tag{2.12}$$

To make use of (2.11) and (2.12) we require the value θ of the anisotropy exponent (ν_P/ν_T). Previous results suggest that $\theta = \log 3/\log 2$ (≈ 1.585), the transfer matrix

calculations of Kinzel and Yeomans (1981) yielding 1.581(1). Preliminary Monte Carlo simulations of our own have confirmed this value, but with much larger error bars. It has also been conjectured that $\theta = \log 3/\log 2$ is an exact result (Kinzel 1983) so we have chosen to use this value in equations (2.11) and (2.12). From (2.8) we see that this particular choice of θ means that we can take $b_P = 3$ and $b_T = 2$ as our rescaling factors.

Similar anisotropic length rescalings for both directed percolation and conductivity problems have been discussed in the literature (Redner and Brooks 1982, Redner and Mueller 1982, Arora *et al* 1983). (We thank one of the referees for drawing our attention to these papers.) The present approach is very similar to that followed by Redner and Mueller (1982) in their Monte Carlo study of the conductivity problem.

We repeat the above Monte Carlo simulations starting with $L_P = 9$ and $L_T = 4$, but now we increase the size of the lattice by a factor of 3 each time in the preferred direction and by a factor of 2 each time in the transverse direction. The results for these anisotropic lattices are shown in table 2. In figure 4 we have plotted $\log[\sigma(b_P L_P, b_P^{1/\theta} L_T)^{-1}]$ against $\log(b_P L_P)$, which according to (2.11) should give a straight line with slope ν_P^{-1}. From the fit to (2.11) (leaving out data for $L_P = 9$, 6561, 531 441) we obtain

$$\nu_P = 1.73(1) \tag{2.13}$$

which is in good agreement with previous results. More importantly, substituting from (2.13) into (2.12) and plotting $\langle P_C(b_P L_P, b_P^{1/\theta} L_T)\rangle$ against $(b_P L_P)^{-1/\nu_P}$ we obtain the expected straight line, as shown in figure 5. This yields an estimate for P_C of

$$P_C = 0.7055(1) \tag{2.14}$$

in excellent agreement with the most recent series estimates.

Table 2. Results for anisotropic finite lattices.

L_P	L_T	Number of realisations	$\langle P_C(L_P, L_T)\rangle$	$\sigma(L_P, L_T)$
9	4	524 288	0.666 31(14)	0.092 83(12)
27	8	524 288	0.681 90(7)	0.049 85(5)
81	16	409 600	0.692 43(4)	0.026 40(3)
243	32	184 320	0.698 48(3)	0.013 90(3)
729	64	40 960	0.701 77(4)	0.007 42(3)
2187	128	4096	0.703 56(6)	0.003 81(6)
6561	256	1536	0.704 37(5)	0.002 01(5)
531 441	4096	20	0.705 40(3)	0.001 13(3)

Our results show the importance of choosing the correct θ. The conjectured value, $\theta = \log 3/\log 2$, is strongly supported by the scaling behaviour seen over almost four decades in figure 5. An analysis of the sensitivity of the results to small changes in θ is difficult due to the need to choose rescaling factors which give a series of integer-sized samples. Nevertheless the results would seem to indicate that $\theta = \log 3/\log 2$, if not an exact result, is at the very least a good approximation.

Regarding this conjecture we note here a possible connection with the statistical mechanics of cellular automata. Wolfram (1983) has shown for one-dimensional

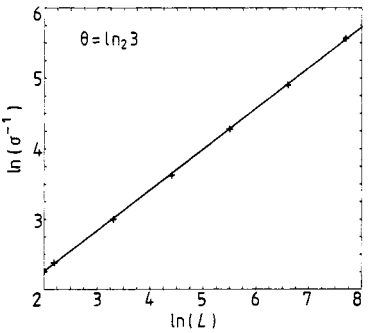

Figure 4. A plot of $\log[\sigma(b_P L_P, b_P^{1/\theta} L_T)^{-1}]$ against $\log[b_P L_P]$ where $\theta = \log 3/\log 2$ and $L_P = 9$, $L_T = 4$. The slope yields ν_P^{-1}.

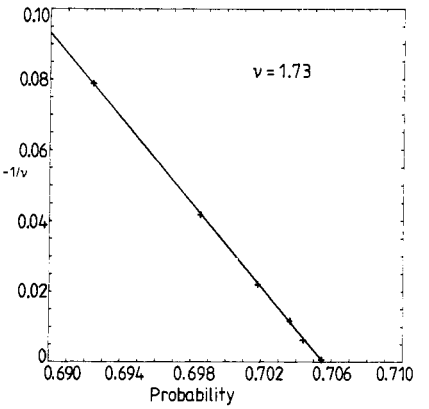

Figure 5. A plot of $(b_P L_P)^{-1/\nu_P}$ against $\langle P_C(b_P L_P, b_P^{1/\theta} L_T)\rangle$ where $\theta = \log 3/\log 2$ and $L_P = 9$, $L_T = 4$. Extrapolating as $b_P L_P \to \infty$ yields P_C.

neighbourhood three cellular automata, that patterns which 'grow' from any simple initial state according to all but one of the 'complex' rules share the universal feature of self-similarity, characterised by a fractal dimension $\log 3/\log 2$. The self-similar structures exhibited in directed percolation arise through local processes which are very similar to cellular automaton rules.

Finally we estimate the exponent β by counting the fraction of sites belonging to the infinite cluster, $F_\infty(P)$, which we expect to behave like

$$F_\infty(P) \sim (P - P_C)^\beta \tag{2.15}$$

as P approaches P_C from above. Following Dhar and Barma (1981) the value of $F_\infty(P)$ is estimated by counting the fraction of percolating sites in the last 500 000 rings of a $4096 \times 640\,000$ cylinder, with the cylinder axis lying along the preferred direction and with every site in the initial ring being occupied. In figure 6 we show a log–log plot of $F_\infty(P)$ against $(P - P_C^{\text{trial}})$, where P_C^{trial} is taken to be 0.7055. This particular trial value for P_C yields the following result for β

$$\beta = 0.2727(7) \qquad (P_C^{\text{trial}} = 0.7055). \tag{2.16}$$

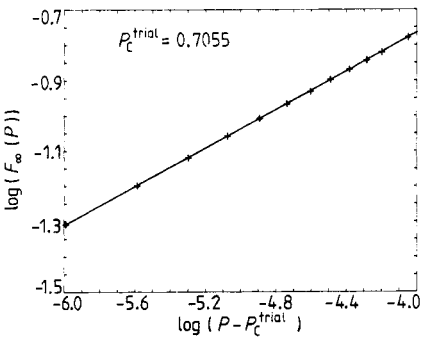

Figure 6. A plot of $\log[F_\infty(P)]$ against $\log(P - P_C^{\text{trial}})$ with $P_C^{\text{trial}} = 0.7055$. The slope yields an estimate of β.

Fits to $P_C^{trial} = 0.7054$ and $P_C^{trial} = 0.7056$ yield

$$\beta = 0.2745(8) \qquad (P_C^{trial} = 0.7054) \tag{2.17a}$$

$$\beta = 0.2707(7) \qquad (P_C^{trial} = 0.7056). \tag{2.17b}$$

These results show the sensitivity to the choice of P_C^{trial}. The difficulty with this method is that the data become less and less reliable the closer P gets to P_C due to the fluctuations becoming larger. This makes it difficult to perform a simultaneous fit to both β and P_C. In fact (2.16) gives the best fit to the data but (2.17*b*) is almost as good.

3. Concluding remarks

In summary we have demonstrated that Monte Carlo methods can be used in directed percolation to yield reliable estimates of the critical threshold probability and exponents, provided that the anisotropy is accounted for correctly. The results obtained compare favourably with those obtained by other methods as shown in table 3. Our results also support the conjecture that $\theta = \log 3/\log 2$ as shown by the scaling behaviour over almost four decades exhibited in figure 5. This conjecture deserves further attention.

In conclusion we note once again that finite-size scaling should be applied with caution to models which exhibit anisotropic divergences of critical correlations.

Table 3. Comparison with previous results.

	P_C	ν_P	ν_T	β
Present results	0.7055(1)	1.73(1)	1.09(1)	0.273(2)
Series[a]	$0.7055(1)+0.02\Delta\gamma$	$1.715(10)+25\Delta P_C$	$1.095(2)+25\Delta P_C$	$0.2725(15)-6\Delta P_C$
Transfer matrix[b]	0.7058(1)	1.730(2)	1.094(1)	
Monte Carlo (Bond)[c]		1.65(6)		
Monte Carlo (Bond)[d]				0.240(6)

[a] de'Bell and Essam (1983a, b). [c] Kertesz and Vicsek (1980).
[b] Kinzel and Yeomans (1981). [d] Dhar and Barma (1981).

4. Acknowledgments

We would like to thank C K Harris for helpful discussions and both the SERC and the ERCC for providing computer time on an ICL Distributed Array Processor. We would also like to thank the SERC for financial support whilst this work was carried out.

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